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2006 J. Phys. A: Math. Gen. 39 7531

(http://iopscience.iop.org/0305-4470/39/23/024)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7531–7542 doi:10.1088/0305-4470/39/23/024

The moving boundary problem in the presence of a dipole magnetic field

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Received 9 November 2005
Published 23 May 2006
Online at stacks.iop.org/JPhysA/39/7531

Abstract
An exact analytic solution is obtained for a uniformly expanding, neutral, infinitely conducting plasma sphere in an external dipole magnetic field. The electrodynamical aspects related to the radiation and transformation of energy were considered as well. The results obtained can be used in analysing the recent experimental and simulation data.

PACS numbers: 03.50.De, 41.20.Gz, 41.20.Jb, 52.30.−q

1. Introduction

Many processes in physics involve boundary surfaces, which requires the solution of boundary and initial value problems. The introduction of a moving boundary into the physics usually precludes the achievement of an exact analytic solution of the problem and recourse to the approximation methods is required [1, 2] (see also [3] and references therein). In the case of a moving plane boundary a time-dependent translation of the embedding space immobilizes the boundary at the expense of the increased complexity of the differential equation. It is the aim of this work to present an example of a soluble moving boundary and initial value problem in the spherical geometry.

The problems with the moving boundary arise in many area of physics. One important example is the sudden expansion of hot plasma with a sharp boundary in an external magnetic field which is particularly of interest for many astrophysical and laboratory applications (see, e.g., [4] and references therein). Such kind of processes arise during the dynamics of solar flares and flow of the solar wind around the earth's magnetosphere, in active experiments with plasma clouds in space, and in the course of interpreting a number of astrophysical observations [3–9]. Researches on this problem are of considerable interest in connection with the experiments on controlled thermonuclear fusion [11] (a recent review [4] summarizes research in this area over the past four decades).

To study the radial dynamics and evolution of the initially spherical plasma cloud both analytical and numerical approaches were developed (see, e.g., [3–9] and references therein). The plasma cloud is shielded from the penetration of the external magnetic field by means of the surface currents circulating inside the thin layer on the plasma boundary. Ponderomotive forces resulting from interaction of these currents with the magnetic field would act on the plasma surface as if there were magnetic pressure applied from outside. After some period of accelerated motion, plasma gets decelerated as a result of this external magnetic pressure acting inward. The plasma has been considered as a highly conducting matter with zero magnetic field inside. From the point of view of electrodynamics it is similar to the expansion of a superconducting sphere in a magnetic field. An exact analytic solution for a uniformly expanding, superconducting plasma sphere in an external uniform and constant magnetic field has been obtained in [12]. The non-relativistic limit of this theory has been used by Raizer [13] to analyse the energy balance (energy radiation and transformation) during the plasma expansion. The similar problem has been considered in [8] for a plasma layer. In the present paper we study the uniform expansion of the superconducting plasma sphere in the presence of a dipole magnetic field. For this geometry we found an exact analytical solution which can be used in analysing the recent experimental and simulation data (see [10] and references therein).

2. Magnetostatic treatment

In this section we first consider the simpler example of a non-relativistic expansion of the plasma sphere ($v \ll c$, where v is the radial velocity of the sphere) in the presence of a dipole magnetic field. Consider the magnetic dipole $\mathbf{p}$ and a superconducting sphere with radius R located at the origin of the coordinate system. The dipole is placed in the position $\mathbf{r}_0$ from the centre of the sphere ($R < r_0$). The orientation of the dipole is given by the angle θ_p between the vectors $\mathbf{p}$ and $\mathbf{r}_0$. Here it is convenient to introduce the scalar magnetic potential $\psi_0(\mathbf{r})$ of the dipole magnetic field which is given by

$$\psi_0(\mathbf{r}) = \frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}. \tag{1}$$

The dipole magnetic field is then calculated as $\mathbf{H}_0(\mathbf{r}) = -\nabla\psi_0(\mathbf{r})$,

$$\mathbf{H}_0(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}_0|^3}\left[\frac{3(\mathbf{r}-\mathbf{r}_0)[\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}_0)]}{|\mathbf{r}-\mathbf{r}_0|^2} - \mathbf{p}\right]. \tag{2}$$

When the superconducting sphere is introduced into a background magnetic field, the plasma expands and excludes the background magnetic field to form a magnetic cavity. The magnetic energy of the dipole in the excluded volume, i.e., in the volume of the superconducting sphere, is calculated as

$$\begin{aligned} Q_R &= \int_{r\leqslant R} \frac{H_0^2(\mathbf{r})}{8\pi}\,\mathrm{d}\mathbf{r} \\ &= \frac{p^2}{32r_0^3}\left\{\frac{\xi(1-\xi^4)(3\cos^2\theta_p - 1) + 8\xi^3(1+\cos^2\theta_p)}{(1-\xi^2)^3} - \frac{3\cos^2\theta_p - 1}{2}\ln\frac{1+\xi}{1-\xi}\right\}, \end{aligned} \tag{3}$$

where $\xi = R/r_0 < 1$. This energy increases with decreasing θ_p and reaches its maximum value at $\theta_p = 0$ or $\theta_p = \pi$, that is the magnetic moment $\mathbf{p}$ is parallel or antiparallel to the symmetry axis $\mathbf{r}_0$. In addition, the magnetic energy Q_R decays rapidly with the distance r_0 and for large $r_0 \gg R$ is given by

$$Q_R = \frac{p^2R^3}{6r_0^6}(3\cos^2\theta_p + 1). \tag{4}$$

In the case when the dipole approaches to the surface of the sphere $r_0 \simeq R$ the magnetic field of the dipole becomes very large and tends to the infinity as

$$Q_R = \frac{p^2}{32r_0^3}\frac{1+\cos^2\theta_p}{(1-\xi)^3}. \tag{5}$$

We now turn to solve the boundary problem and calculate the induced magnetic field which arises near the surface of the sphere due to the dipole magnetic field. Since the sphere is superconducting the magnetic field vanishes inside the sphere. In addition, the normal component of the field H_r vanishes on the surface of the sphere. To solve the boundary problem we introduce the spherical coordinate system with the z-axis along the vector $\mathbf{r}_0$ and the azimuthal angle ϕ is counted from the plane (xz-plane) containing the vectors $\mathbf{r}_0$ and $\mathbf{p}$. Hence, using expressions (A.2)–(A.4), the scalar potential (1) at $r < r_0$ can alternatively be represented by the sum of Legendre polynomials (see appendix A for details):

$$\psi_0(\mathbf{r}) = \frac{p}{r_0^2}\left[\sin\theta_p\cos\phi\sum_{l=1}^{\infty}\left(\frac{r}{r_0}\right)^l P_l^1(\cos\theta) - \cos\theta_p\sum_{l=0}^{\infty}(l+1)\left(\frac{r}{r_0}\right)^l P_l(\cos\theta)\right]. \tag{6}$$

The total magnetic field which is a sum of $\mathbf{H}_0(\mathbf{r})$ and the induced magnetic field is obtained from the equation $\nabla\cdot\mathbf{H} = 0$. Introducing the scalar potential, $\mathbf{H}(\mathbf{r}) = -\nabla\psi(\mathbf{r})$, the last equation becomes $\nabla^2\psi(\mathbf{r}) = 0$, i.e., $\psi(\mathbf{r})$ satisfies the Laplace equation. We must solve this equation with $\mathbf{H} = 0$ at $r < R$ and the boundary condition

$$H_r|_{r=R} = -\left.\frac{\partial\psi}{\partial r}\right|_{r=R} = 0. \tag{7}$$

We look for the solution of the Laplace equation which in a spherical coordinate system and at $r \geqslant R$ can be written as

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \frac{p}{r_0^2}\left[\sum_{l=0}^{\infty}\alpha_l\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta) + \cos\phi\sum_{l=1}^{\infty}\beta_l\left(\frac{R}{r}\right)^{l+1} P_l^1(\cos\theta)\right], \tag{8}$$

where α_l and β_l are the arbitrary constants and should be obtained from the boundary condition (7). The second term in equation (8) is the induced magnetic field. From equations (6)–(8) one finds

$$\alpha_l = -l\left(\frac{R}{r_0}\right)^l\cos\theta_p, \qquad \beta_l = \frac{l}{l+1}\left(\frac{R}{r_0}\right)^l\sin\theta_p. \tag{9}$$

Substituting equation (9) into equation (8) and using the summation formula obtained in appendix A from (8) we find

$$\psi(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{R}_0}{R_0^3} + \frac{\mathbf{Q}\cdot\mathbf{R}_*}{R_*^3} + \psi_{\mathrm{QD}}(\mathbf{r}), \tag{10}$$

where

$$\psi_{\mathrm{QD}}(\mathbf{r}) = -\xi^3\frac{(\mathbf{p}_\perp\cdot\mathbf{R}_*)}{R_*^3}\left(\frac{R_*^2}{\mathbf{r}\cdot\mathbf{R}_* + rR_*} - \frac{1}{2}\right). \tag{11}$$

Here $\mathbf{r}_* = \xi^2\mathbf{r}_0$, $\mathbf{R}_0 = \mathbf{r} - \mathbf{r}_0$, $\mathbf{R}_* = \mathbf{r} - \mathbf{r}_*$,

$$\mathbf{p}_\perp = \mathbf{p} - \frac{(\mathbf{p}\cdot\mathbf{r}_0)\mathbf{r}_0}{r_0^2}, \qquad \mathbf{Q} = \frac{\xi^3}{2}\left[\mathbf{p} - \frac{3(\mathbf{p}\cdot\mathbf{r}_0)\mathbf{r}_0}{r_0^2}\right]. \tag{12}$$

The term $\psi_{\mathrm{QD}}(\mathbf{r})$ in equation (10) can be interpreted as a magnetic field of point-like quadrupole with the 'quadrupole moment' $D_{\alpha\beta}(\mathbf{r})$ and located in the xz-plane inside the sphere at the distance $\mathbf{r}_*(r_* = \xi R < R)$ from the centre. At large distances this term behaves as $\psi_{\mathrm{QD}}(\mathbf{r}) \simeq xzD_{xz}/r^5$ with the quadrupole moment $D_{xz} = \frac{r_0}{2}\xi^5 p\sin\theta_p$ ($D_{\alpha\alpha} = D_{xy} = D_{yz} = 0$ and $\alpha = x, y, z$). The induced electric field is calculated from Maxwell's equation $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$. However, if the plasma radial velocity is small, $v/c \ll 1$, the amplitude of the electric field is small as well (of the order of $\frac{v}{c}H_0(\mathbf{r})$) and may be completely ignored. Below we consider two particular cases for the magnetic dipole orientation in the space.

(i) *The case* $\theta_p = 0; \pi$. In this case the magnetic dipole is parallel or antiparallel to the vector $\mathbf{r}_0$. Obviously due to the symmetry reason the magnetic field does not depend on ϕ and $H_\phi = 0$. The magnetic field component $H_\theta = -(1/r)(\partial\psi/\partial\theta)$ induces the surface current on the sphere. The ponderomotive forces resulting from the interaction of this current with the magnetic field act on the sphere surface with a magnetic pressure which can be calculated as an energy density of the magnetic field,

$$P_{\parallel}(\theta) = \left.\frac{H_\theta^2}{8\pi}\right|_{r=R} = \frac{9p^2}{8\pi r_0^6}\frac{(1-\xi^2)^2\sin^2\theta}{(\xi^2+1-2\xi\cos\theta)^5}. \tag{13}$$

This pressure vanishes at $\theta = 0, \pi$ and has its maximum at

$$\cos\theta_{\max} = \frac{10\xi}{\sqrt{(\xi^2+1)^2+60\xi^2}+\xi^2+1}. \tag{14}$$

The value of $\theta_{\max}$ tends to zero when the dipole comes close to the sphere and shifts towards the larger values, $\theta_{\max} \simeq \pi/2$, when the dipole goes to the infinity. Therefore the layer near $\theta \simeq \theta_{\max}$ of the expanding sphere will be mainly deformed by the external magnetic pressure. This behaviour is clearly seen in the particle-in-cell simulation [14].

The total force is calculated as a surface integral of the magnetic pressure,

$$\mathcal{F}_{\parallel} = 2\pi R^2 \int_0^\pi P_{\parallel}(\theta)\sin\theta\,\mathrm{d}\theta = \frac{3p^2}{r_0^4}\frac{\xi^2(1+\xi^2)}{(1-\xi^2)^4}. \tag{15}$$

This force behaves as $\mathcal{F}_{\parallel} \sim l^{-s}$ with $s = 6$ and $s = 4$ at large and small distances between the dipole and the surface of the sphere, respectively.

(ii) *The case* $\theta_p = \pi/2$. In this case there are two components of the surface currents which are proportional to H_θ and H_ϕ at $r = R$. The magnetic pressure is then given by

$$P_{\perp}(\theta,\phi) = \left.\frac{H_\theta^2 + H_\phi^2}{8\pi}\right|_{r=R} = \frac{p^2}{8\pi r_0^6}\frac{\Upsilon_1^2(\xi,\theta)\cos^2\phi + \Upsilon_2^2(\xi,\theta)\sin^2\phi}{\Upsilon^6(\xi,\theta)}, \tag{16}$$

where

$$\Upsilon_1(\xi,\theta) = \Upsilon_2(\xi,\theta)\cos\theta - \xi\sin^2\theta\left[\frac{6}{\Upsilon^2} - \frac{1}{1-\xi\cos\theta+\Upsilon} - \frac{\Upsilon(1+\Upsilon)}{(1-\xi\cos\theta+\Upsilon)^2}\right], \tag{17}$$

$$\Upsilon_2(\xi,\theta) = \frac{1-\xi^2+2\Upsilon}{1-\xi\cos\theta+\Upsilon}, \qquad \Upsilon = \sqrt{1+\xi^2-2\xi\cos\theta}. \tag{18}$$

At large distances, $\xi \ll 1$, the magnetic pressure is maximum at $\phi \simeq \frac{\pi}{2}$ and $\frac{3\pi}{2}$ (in equatorial plane), and $\theta = 0, \pi$. At small distances, $1 - \xi \ll 1$, only the region of the sphere with $\theta \sim 1 - \xi \sim 0$ will be strongly deformed.

The total ponderomotive magnetic force acting on the sphere is calculated as

$$\mathcal{F}_{\perp} = R^2\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^{2\pi} P_{\perp}(\theta,\phi)\,\mathrm{d}\phi = \frac{p^2}{4r_0^4}\frac{\xi^2(3+8\xi^2+\xi^4)}{(1-\xi^2)^4}. \tag{19}$$

Again as for $\theta_p = 0, \pi$ the force $\mathcal{F}_\perp$ behaves as $\mathcal{F}_\perp \sim l^{-s}$ with $s = 6$ and $s = 4$ at large and small distances, respectively. However, comparing equations (15) and (19) we conclude that the total magnetic force at $\theta_p = \pi/2$ is smaller than for parallel or antiparallel orientation of the dipole. For instance, from equations (15) and (19) we obtain $\mathcal{F}_\parallel \simeq 4\mathcal{F}_\perp$ and $\mathcal{F}_\parallel \simeq 2\mathcal{F}_\perp$ at $\xi \ll 1$ and $\xi \sim 1$, respectively.

3. Electrodynamic treatment

In this section we consider the moving boundary problem of the plasma sphere expansion in the vacuum. In this sense unlike the magnetostatic problem considered above it is convenient here to introduce the vector potential of the induced and dipole magnetic fields. Consider a spherical region of space containing a neutral infinitely conducting plasma which has expanded at $t = 0$ to its present state from a point source located at the point $\mathbf{r} = 0$. The external space at the point $\mathbf{r}_0$ contains a magnetic dipole $\mathbf{p}$. The magnetic field of this dipole is given by $\mathbf{H}_0 = \nabla \times \mathbf{A}_0$, where the vector potential $\mathbf{A}_0$ is

$$\mathbf{A}_0 = \frac{\mathbf{p} \times (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}. \tag{20}$$

As the spherical plasma cloud expands it both perturbs the external magnetic field and generates an electric field. Within the spherical plasma region there is neither an electric field nor a magnetic field. We shall obtain an analytic solution of the electromagnetic field configuration.

We consider a practically interesting case when the vectors $\mathbf{p}$ and $\mathbf{r}_0$ are parallel (or antiparallel). The general solution for the arbitrary orientation of $\mathbf{p}$ will be considered in a separate paper. Within this geometry the problem is symmetric with respect to the axis $\mathbf{r}_0$ which is chosen as the axial axis of the spherical coordinate system. Then there is only one nonvanishing component of $\mathbf{A}_0$, $A_{0r} = A_{0\theta} = 0$, and

$$A_{0\varphi} = \frac{pr\sin\theta}{|\mathbf{r} - \mathbf{r}_0|^3} = \frac{p}{r_0^2} \sum_{l=1}^{\infty} D_l\left(\frac{r}{r_0}\right) P_l^1(\cos\theta), \tag{21}$$

where $P_l^\nu(x)$ is the generalized Legendre polynomials with $\nu = 1$. Here $D_l(x) = x^l$ at $x \leqslant 1$ and $D_l(x) = x^{-l-1}$ at $x > 1$ as defined in appendix A.

Since the external region is devoid of free charge density, a suitable gauge allows the electric and magnetic fields to be derived from the vector potential $\mathbf{A}$. Having in mind the symmetry of the original dipole magnetic field it is sufficient to choose the vector potential in the form $A_r = A_\theta = 0$,

$$A_\varphi(r, \theta, t) = A_{0\varphi}(r, \theta) + \sum_{l=1}^{\infty} \mathcal{A}_l(r, t) P_l^1(\cos\theta), \tag{22}$$

and the components of the electromagnetic field are given by

$$H_r = \frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}, \qquad H_\theta = -\frac{\partial A_\varphi}{\partial r}, \qquad E_\varphi = -\frac{1}{c}\frac{\partial A_\varphi}{\partial t}, \tag{23}$$

and $H_\varphi = E_r = E_\theta = 0$. The equation for $\mathcal{A}_l(r, t)$ is obtained from Maxwell's equations,

$$\frac{\partial^2 \mathcal{A}_l}{\partial r^2} + \frac{2}{r}\frac{\partial \mathcal{A}_l}{\partial r} - \frac{l(l+1)}{r^2}\mathcal{A}_l - \frac{1}{c^2}\frac{\partial^2 \mathcal{A}_l}{\partial t^2} = 0. \tag{24}$$

This equation is to be solved in the external region $r > R(t)$ subject to the boundary and initial conditions. Here $R(t)$ is the plasma sphere radius at the time t. The initial conditions are at $t = 0$,

$$\mathcal{A}_l(r, 0) = 0, \qquad \frac{\partial \mathcal{A}_l(r, 0)}{\partial t} = 0. \tag{25}$$

The first initial condition states that the initial value of A_φ is that of a dipole magnetic field. The second initial condition states that there is no initial electric field. Boundary conditions should be imposed at the spherical surface $r = R(t)$ and at infinity. Because of the finite propagation velocity of the perturbed electromagnetic field the magnetic field at infinity will remain undisturbed for all finite times. Further, no incoming wave-type solutions are permitted. Thus, for all finite times $\mathcal{A}_l(r,t) \to 0$ at $r \to \infty$. The boundary condition at the expanding spherical surface is $H_r = 0$, which can be replaced by $A_\varphi(R(t),\theta,t) = 0$ or, alternatively,

$$\mathcal{A}_l(R(t),t) = -\frac{p}{r_0^2} D_l\left(\frac{R(t)}{r_0}\right). \tag{26}$$

The problem of solving equation (24) subject to the initial and boundary conditions will be accomplished by the Laplace transform theory. The Laplace transform $\widetilde{\mathcal{A}}_l(r,\lambda)$ of the function $\mathcal{A}_l(r,t)$ is introduced by

$$\widetilde{\mathcal{A}}_l(r,\lambda) = \int_0^\infty \mathcal{A}_l(r,t)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}t \tag{27}$$

with $\operatorname{Re}\lambda > 0$. An inverse transformation is established by

$$\mathcal{A}_l(r,t) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \widetilde{\mathcal{A}}_l(r,\lambda)\,\mathrm{e}^{\lambda t}\,\mathrm{d}\lambda. \tag{28}$$

The real parameter σ should be larger than $\operatorname{Re}\lambda_i$, $\sigma > \operatorname{Re}\lambda_i$, where λ_i are the poles of $\widetilde{\mathcal{A}}_l(r,\lambda)$.

The differential equation for $\widetilde{\mathcal{A}}_l(r,\lambda)$ is found from equations (24) and (28) and the initial conditions in (25):

$$\frac{\partial^2 \widetilde{\mathcal{A}}_l(r,\lambda)}{\partial r^2} + \frac{2}{r}\frac{\partial \widetilde{\mathcal{A}}_l(r,\lambda)}{\partial r} - \left[\frac{l(l+1)}{r^2} + \frac{\lambda^2}{c^2}\right]\widetilde{\mathcal{A}}_l(r,\lambda) = 0. \tag{29}$$

Its solution may be written as

$$\widetilde{\mathcal{A}}_l(r,\lambda) = \frac{p}{r_0^2}\left[a_l(\lambda) h_l^{(1)}\left(\mathrm{i}\frac{\lambda}{c}r\right) + c_l(\lambda) h_l^{(2)}\left(\mathrm{i}\frac{\lambda}{c}r\right)\right], \tag{30}$$

where $h_l^{(1)}(z)$ and $h_l^{(2)}(z)$ are the Hankel spherical functions and $a_l(\lambda), c_l(\lambda)$ are arbitrary functions of λ determined from the boundary conditions. Since $h_l^{(2)}(z)$ gives rise to incoming waves, we should set $c_l(\lambda) = 0$. The solution to equation (24) at $r > R(t)$ now may be written in the form

$$A_\varphi(r,\theta,t) = \frac{p}{r_0^2}\sum_{l=1}^{\infty} P_l^1(\cos\theta)\left[D_l\left(\frac{r}{r_0}\right) + \frac{1}{2\pi}\int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} b_l(\lambda) h_l^{(1)}\left(\frac{\lambda}{c}r\right)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}\lambda\right], \tag{31}$$

where $b_l(\lambda) = a_l(-\mathrm{i}\lambda)$.

The moving boundary condition in equation (26) requires the satisfaction of

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} b_l(\lambda) h_l^{(1)}\left(\frac{\lambda}{c}R(t)\right)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}\lambda = \mathrm{i}D_l\left(\frac{R(t)}{r_0}\right). \tag{32}$$

Since the sphere moves with a radial velocity v less than the velocity of light c, we have $R < ct$ or $t - R(t)/c > 0$. Thus, the contour in the integral of equation (32) should be closed by an infinite semicircle in the lower half plane and the integral evaluated by the method of residues.

Explicit evaluation of this integral equation (32) may be accomplished in the special case of a uniform expansion. Choosing the simple model of constant radial velocity $R(t) = vt$ and assuming that $R(t) < r_0$ equation (32) yields (see appendix B for details)

$$b_l(\lambda) = \frac{(-1)^l (v/r_0)^l}{\lambda^{l+1}} \frac{\mathrm{i}\beta}{(1-\beta^2)^{\frac{l+1}{2}}} \frac{1}{P_l^{-l-1}(1/\beta)}, \tag{33}$$

where $\beta = v/c < 1$. Here $P^{\nu}_{\mu}(z)$ are the generalized Legendre functions with $z > 1$, $\mu = l$ and $\nu = -l - 1$.

The solution of equations (24) and (31) may be obtained by inserting equation (33) into (31) and evaluating the integral (see appendix B for details). The complete solution may finally be written in the form, at $vt < r < ct$,

$$A_{\varphi}(r, \theta, t) = A_{0\varphi}(r, \theta) - \frac{p}{r_0^2} \sum_{l=1}^{\infty} \left(\frac{r}{r_0}\right)^l \frac{p_l(1/\zeta)}{p_l(1/\beta)} P_l^1(\cos\theta), \tag{34}$$

$A_{\varphi}(r, \theta, t) = A_{0\varphi}(r, \theta)$ at $r \geqslant ct$ and $A_{\varphi}(r, \theta, t) = 0$ at $r \leqslant vt$. Here $\zeta = r/ct < 1$, and

$$p_l(z) = 2^l l!(z^2 - 1)^{\frac{l+1}{2}} P_l^{-l-1}(z) = \int_1^z (\tau^2 - 1)^l \, \mathrm{d}\tau. \tag{35}$$

The electromagnetic field components can be evaluated according to equation (23). From equations (23) and (34) it can be easily checked that the boundary condition on the moving surface, $\mathbf{E}(R) = -\frac{1}{c}[\mathbf{v} \times \mathbf{H}(R)]$ (or $E_{\varphi}(R) = -\beta H_{\theta}(R)$), is satisfied automatically. It may also be noted that this special case of the uniform expansion falls within the conical flow techniques, as indicated in [12] for the case of a uniform magnetic field. From symmetry considerations one seeks a solution of the form $\mathcal{A}_l(r, t) = r^{\nu}\Phi(r/ct)$. Substitution into the differential equation (24) yields an explicitly solvable ordinary differential equation whose solution, upon application of the boundary conditions $\left(\Phi(1) = 0,\ \Phi(\beta) = -p/r_0^{l+2}\right)$, is given by equation (34).

It should be noted that all the above results are valid only for $R(t) < r_0$ or $t < r_0/v$. At the time $t = r_0/v$ the dipole will enter into the plasma sphere and hence will be completely shielded by the latter. Therefore at $t \geqslant r_0/v$ the total electromagnetic field vanishes and the radiation is interrupted.

4. Energy balance

Previously significant attention has been paid [8, 13] to the question of what fraction of energy is emitted and lost in the form of electromagnetic pulse propagating outwards of the expanding plasma. In this section we consider the energy balance during the plasma sphere expansion in the presence of the magnetic dipole. When the plasma sphere of the zero initial radius is created at $t = 0$ and starts expanding, external magnetic field $\mathbf{H}_0$ is perturbed by the electromagnetic pulse, $\mathbf{H}'(\mathbf{r}, t) = \mathbf{H}(\mathbf{r}, t) - \mathbf{H}_0(\mathbf{r})$, $\mathbf{E}(\mathbf{r}, t)$, propagating outwards with the speed of light. The tail of this pulse coincides with the moving plasma boundary $r = R(t)$ while the leading edge is at $r = ct$. Ahead of the leading edge, the magnetic field is not perturbed and equals $\mathbf{H}_0(\mathbf{r})$ while the electric field is zero.

Our starting point is the energy balance equation (Poynting equation)

$$\nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} - \frac{\partial}{\partial t} \frac{E^2 + H^2}{8\pi}, \tag{36}$$

where $\mathbf{S} = \frac{c}{4\pi}[\mathbf{E} \times \mathbf{H}]$ is the Poynting vector and $\mathbf{j} = j_{\varphi}\mathbf{e}_{\varphi}$ (with $|\mathbf{e}_{\varphi}| = 1$) is the azimuthal surface current density. The energy radiated to infinity is measured as a Poynting vector integrated over time and over the surface S_c of the sphere with radius $r_c < r_0$ (control sphere) and the volume Ω_c enclosing the plasma sphere ($r_c > R$ or $0 \leqslant t < r_c/v$). Integrating over time and over the volume Ω_c equation (36) can be represented as

$$W_S(t) = W_J(t) + \Delta W_{\mathrm{EM}}(t), \tag{37}$$

where

$$W_S(t) = 2\pi r_c^2 \int_0^t \mathrm{d}t' \int_0^{\pi} S_r \sin\theta \, \mathrm{d}\theta, \qquad W_J(t) = -\int_0^t \mathrm{d}t' \int_{\Omega_c} \mathbf{j} \cdot \mathbf{E} \, \mathrm{d}\mathbf{r}. \tag{38}$$

Here $S_r = -\frac{c}{4\pi}E_{\varphi}H_{\theta}$ is the radial component of the Poynting vector. $W_{\mathrm{EM}}(t)$ and $\Delta W_{\mathrm{EM}}(t) = W_{\mathrm{EM}}(0) - W_{\mathrm{EM}}(t)$ are the total electromagnetic energy and its change (with minus sign) in a volume Ω_c, respectively. $W_J(t)$ is the energy transferred from the plasma sphere to the electromagnetic field and is the mechanical work with minus sign performed by the plasma on the external electromagnetic pressure. At $t = 0$ the electromagnetic fields are given by $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0(\mathbf{r})$ and $\mathbf{E}(\mathbf{r}, t) = 0$. Hence $W_{\mathrm{EM}}(0)$ is the energy of the dipole magnetic field in a volume Ω_c and can be calculated from equation (3) by replacing R by r_c and setting $\sin\theta_p = 0$,

$$W_{\mathrm{EM}}(0) = \int_{\Omega_c} \frac{H_0^2(\mathbf{r})}{8\pi}\,\mathrm{d}\mathbf{r} = Q(u) = \frac{p^2}{16r_0^3}\left[\frac{u(1-u^4+8u^2)}{(1-u^2)^3} - \frac{1}{2}\ln\frac{1+u}{1-u}\right], \tag{39}$$

where $u = \frac{r_c}{r_0} < 1$. Then the change of the electromagnetic energy $\Delta W_{\mathrm{EM}}(t)$ in a volume Ω_c can be evaluated as

$$\Delta W_{\mathrm{EM}}(t) = -\int_{\Omega_c} \frac{E^2 + H^2 - H_0^2}{8\pi}\,\mathrm{d}\mathbf{r} = Q(u) - \int_{\Omega_c'} \frac{E^2 + H^2}{8\pi}\,\mathrm{d}\mathbf{r}. \tag{40}$$

In equation (40) Ω_c' is the volume of the control sphere excluding the volume of the plasma sphere (we take into account that $\mathbf{H}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) = 0$ in a plasma sphere). Hence the total energy flux $W_S(t)$ given by equation (38) is calculated as a sum of the energy loss by the plasma due to the external electromagnetic pressure and the decrease of the electromagnetic energy in a control volume Ω_c. For the non-relativistic ($\beta \ll 1$) expansion of a one-dimensional plasma slab and for a uniform external magnetic field ($\mathbf{H}_0 = \mathrm{const}$) $W_S \simeq 2W_J \simeq 2\Delta W_{\mathrm{EM}}$, i.e., approximately the half of the outgoing energy is gained from the plasma, while the other half is gained from the magnetic energy [8]. In the case of the non-relativistic expansion of a highly-conducting spherical plasma in the uniform magnetic field the outgoing energy W_S is distributed between W_J and ΔW_{EM} according to $W_J = 1.5Q_0$ and $\Delta W_{\mathrm{EM}} = 0.5Q_0$ with $W_S = 2Q_0$, where $Q_0 = H_0^2R^3/6$ is the magnetic energy escaped from the plasma volume [13]. Therefore in this case the released electromagnetic energy is mainly gained from the plasma.

Consider now each energy component $W_S(t)$, $W_J(t)$ and $\Delta W_{\mathrm{EM}}(t)$ separately. $W_S(t)$ is calculated from equation (38). In the first expression of equation (38) the t'-integral must be performed at $\frac{r_c}{c} \leqslant t' \leqslant t$ ($t < \frac{r_c}{v}$) since at $0 \leqslant t' < \frac{r_c}{c}$ the electromagnetic pulse does not reach to the control surface yet and $S_r(r_c) = 0$. From equations (23), (34) and (38) we obtain

$$W_S(t) = Q(u) + \frac{p^2}{2r_0^3}\sum_{l=1}^{\infty}\frac{l(l+1)}{2l+1}u^{2l+1}\left\{\frac{(1/\eta^2-1)^{2l+1}}{(2l+1)p_l^2(1/\beta)} - (l+1)\left[\frac{p_l(1/\eta)}{p_l(1/\beta)} - 1\right]^2\right\}, \tag{41}$$

where $\eta = r_c/ct < 1$. In the non-relativistic limit, $\beta \to 0$, using the asymptotic expression (see, e.g., [15]) $p_l(z) = z^{2l+1}/(2l+1)$ at $z \to \infty$, from equation (41) we obtain

$$\begin{aligned} W_S(t) &= 2Q(\xi) - Q(\kappa) + \frac{p^2}{r_0^3}\frac{\kappa^3}{(1-\kappa^2)^3} \\ &= \frac{p^2}{16r_0^3}\left[\frac{2\xi(1+8\xi^2-\xi^4)}{(1-\xi^2)^3} + \frac{\kappa(\kappa^4+8\kappa^2-1)}{(1-\kappa^2)^3} - \frac{1}{2}\ln\frac{(1-\kappa)(1+\xi)^2}{(1+\kappa)(1-\xi)^2}\right] \end{aligned} \tag{42}$$

with $\kappa = R^2/r_0r_c$. In equation (42) $Q(\kappa)$ represents the magnetic energy of the dipole field in a sphere having the radius $R_* = R^2/r_c < R$ and enclosed in the plasma sphere.

Next, we calculate the energy loss $W_J(t)$ by the plasma which is determined by the surface current density, $\mathbf{j}$. From the symmetry reason it is clear that this current has only an azimuthal component and is localized within a thin spherical skin layer, $R-\delta < r < R+\delta$ with $\delta \to 0$, near the plasma boundary. Therefore in equation (38) the volume Ω_c can be replaced by the volume $\Omega_\delta \sim R^2\delta$ which includes the space between the spheres with $r = R-\delta$ and $r = R+\delta$. The surface current density is calculated from Maxwell's equation, $\mathbf{j} = (1/4\pi)\big(c\nabla\times\mathbf{H}-\frac{\partial\mathbf{E}}{\partial t}\big)$. Within the skin layer we take into account that $\mathbf{E} = -\frac{1}{c}[\mathbf{v}\times\mathbf{H}]$ and $H_r(R) = 0$. Then

$$Q_J(t) = -\int_{\Omega_\delta}\mathbf{j}\cdot\mathbf{E}\,\mathrm{d}\mathbf{r} = \frac{1}{4\pi}\int_{\Omega_\delta}\mathbf{v}\cdot[\mathbf{H}\times(\nabla\times\mathbf{H})]\,\mathrm{d}\mathbf{r} + \frac{1}{8\pi}\int_{\Omega_\delta}\frac{\partial\mathbf{E}^2}{\partial t}\,\mathrm{d}\mathbf{r}$$
$$= v\int_{S_R}\frac{H_\theta^2(R)-E_\varphi^2(R)}{8\pi}\,\mathrm{d}S = \frac{v}{\gamma^2}\int_{S_R}\frac{H_\theta^2(R)}{8\pi}\,\mathrm{d}S, \tag{43}$$

where $\gamma^{-2} = 1-\beta^2$ and S_R are the relativistic factor and the surface of the expanding plasma, respectively. Note that the moving boundary modifies the surface current which is now proportional to γ^{-2} [3]. In equation (43) the term with $\frac{\partial\mathbf{E}^2(\mathbf{r},t)}{\partial t}$ has been transformed to the surface integral using the fact that the boundary of the volume Ω_δ moves with a constant velocity v and the electrical field has a jump across the plasma surface. Equation (43) shows that the energy loss by the plasma per unit time is equal to the work performed by the plasma on the external electromagnetic pressure. This external pressure is formed by the difference between magnetic and electric pressures, i.e., the induced electric field tends to decrease the force acting on the expanding plasma surface. The total energy loss by the plasma sphere is calculated as

$$W_J(t) = \int_0^t Q_J(t')\,\mathrm{d}t' = \frac{p^2}{2r_0^3}\sum_{l=1}^{\infty}\frac{l(l+1)}{(2l+1)^2}\left(\frac{\xi}{\beta^2\gamma^2}\right)^{2l+1}\frac{1}{p_l^2(1/\beta)}, \tag{44}$$

where $\xi = R/r_0$. In a non-relativistic case equation (44) yields

$$W_J(t) = \frac{p^2}{r_0^3}\frac{\xi^3}{(1-\xi^2)^3}. \tag{45}$$

The change of the electromagnetic energy in a control sphere is calculated from equation (40). At $R < r_c < ct$ (the electromagnetic pulse fills the whole control sphere) we obtain

$$\Delta W_{\mathrm{EM}}(t) = Q(u) - \frac{p^2}{2r_0^3}\sum_{l=1}^{\infty}\frac{l(l+1)}{(2l+1)^2}\left(\frac{\xi}{\beta^2\gamma^2}\right)^{2l+1}\frac{1}{p_l^2(1/\beta)}$$
$$+\frac{p^2}{2r_0^3}\sum_{l=1}^{\infty}\frac{l(l+1)}{2l+1}u^{2l+1}\left\{\frac{(1/\eta^2-1)^{2l+1}}{(2l+1)p_l^2(1/\beta)} - (l+1)\left[\frac{p_l(1/\eta)}{p_l(1/\beta)}-1\right]^2\right\}. \tag{46}$$

Comparing equations (41), (44) and (46) we conclude that $\Delta W_{\mathrm{EM}}(t) + W_J(t) = W_S(t)$ as predicted by the energy balance equation (37). The non-relativistic limit of equation (46) can be evaluated from equations (42) and (45) using the relation $\Delta W_{\mathrm{EM}}(t) = W_S(t) - W_J(t)$. As an example in figure 1 we show the results of model calculations for the ratios $\Gamma_S(t) = W_S(t)/Q_0(t)$ and $\Gamma_J(t) = W_J(t)/Q_0(t)$ as a function of time ($r_c/c \leqslant t < r_c/v$). Here $Q_0(t) = Q(\xi)$ is the dipole magnetic energy escaped from the plasma sphere. For the relativistic factor β we have chosen a wide range of values. We recall that at $0 \leqslant t \leqslant r_c/c$, i.e. the electromagnetic pulse does not yet reach to the surface of the control sphere, $W_S(t) = 0$. Unlike the case with uniform magnetic field discussed above (see also [8, 13]) there are no simple relations between the energy components $W_S(t)$, $W_J(t)$ and $Q_0(t)$. However, at the initial stage ($t \ll r_c/v$) of the non-relativistic expansion the dipole field at large distances can be treated as uniform and the energies $W_S(t)$ and $W_J(t)$ are close to the values $2Q_0(t)$

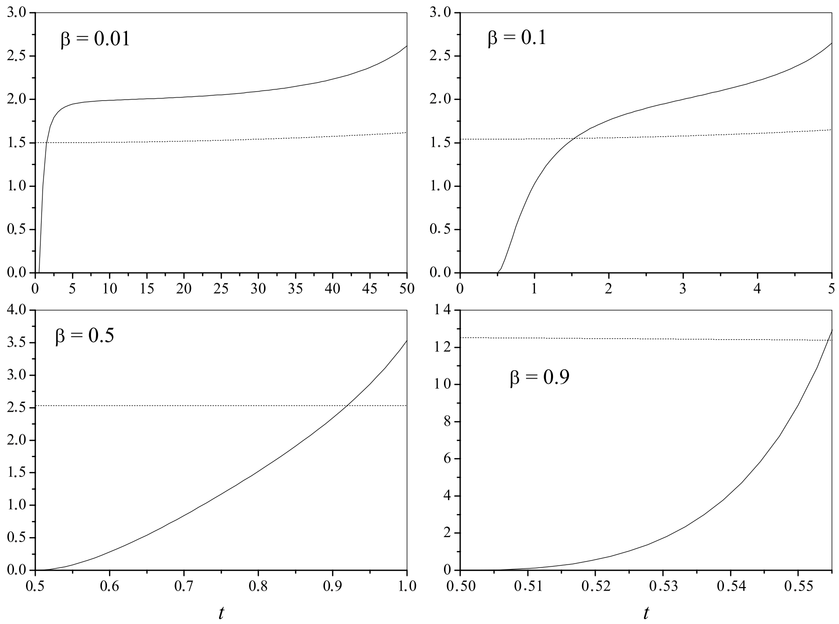

Figure 1. The ratios $\Gamma_S(t)$ (solid lines) and $\Gamma_J(t)$ (dashed lines) for four values of β as a function of t (in units of r_0/c) calculated from expressions (41) and (44) with $r_c = 0.5r_0$.

and $1.5Q_0(t)$ (see figure 1), respectively. For any β the ratio $\Gamma_J(t)$ is almost constant and may be approximated as $\Gamma_J(t) \simeq \Gamma_J(0)$ or alternatively $W_J(t) \simeq 1.5CQ_0(t)$, where $C = \gamma^{-6}(1-\beta)^{-4}(1+2\beta)^{-2}$ is some kinematic factor. For $\beta \sim 1$ this factor is very large and behaves as $C \simeq (8/9)(1-\beta)^{-1} \gg 1$. As expected, the total energy flux, $W_S(t)$, increases monotonically with t. At the final stage ($t = r_c/v$) of the relativistic expansion (with $\beta \sim 1$) $W_S \simeq W_J$. Hence in this case the radiated energy W_S is mainly gained from the plasma sphere.

5. Conclusion

An exact solution of the uniform radial expansion of a neutral, infinitely conducting plasma sphere in the presence of a dipole magnetic field has been obtained. The electromagnetic fields are derived by using the appropriate boundary and initial conditions, equations (25) and (26). It is shown that the electromagnetic fields are perturbed only within the domain extending from the surface of the expanding plasma sphere $r = R = vt$ to the surface of the expanding information sphere $r = ct$. External to the sphere $r = ct$ the magnetic field is not perturbed and is given by the dipole magnetic field. In the course of this study we have also considered the energy balance during the plasma sphere expansion. The model calculations show that the radiated energy is mainly gained from the plasma sphere. For relativistic expansion the ratio W_S/W_J is close to unity and the radiated energy is practically gained only from the plasma sphere.

We expect our theoretical findings to be useful in experimental investigations as well as in numerical simulations of the plasma expansion into an ambient nonuniform magnetic field. One of the improvements of our model will be to include the effect of the deceleration of the plasma sphere as well as the derivation of the dynamical equation for the surface deformation. A study of this and other aspects will be reported elsewhere.

Acknowledgments

This work has been partially supported by the Armenian Ministry of Higher Education and Science Grant. The authors thank the referees for helpful suggestions and comments.

Appendix A. Sums with Legendre polynomials

Using the known relation [15]

$$F_0(x,\theta) = \frac{1}{(1+x^2-2x\cos\theta)^{1/2}} = \sum_{l=0}^{\infty} D_l(x) P_l(\cos\theta), \tag{A.1}$$

where $D_l(x) = x^l$ at $|x| \leqslant 1$ and $D_l(x) = x^{-l-1}$ at $|x| > 1$, one can derive some sums with Legendre polynomials $P_l(\cos\theta)$ which are used in the main text of the paper. The first relation is obtained from equation (A.1) by taking the partial derivative of the function $F_0(x,\theta)$:

$$\frac{\partial}{\partial x} F_0(x,\theta) = \frac{\cos\theta - x}{(1+x^2-2x\cos\theta)^{3/2}} = \sum_{l=0}^{\infty} D_l'(x) P_l(\cos\theta). \tag{A.2}$$

Here the prime indicates the derivative with respect to the argument.

The second relation follows from equation (A.1) if we take the partial derivative over θ:

$$-\frac{\partial}{\partial\theta} F_0(x,\theta) = \frac{x\sin\theta}{(1+x^2-2x\cos\theta)^{3/2}} = \sum_{l=1}^{\infty} D_l(x) P_l^1(\cos\theta), \tag{A.3}$$

where $P_l^1(\cos\theta)$ are the generalized Legendre polynomials $P_l^{\nu}(\cos\theta)$ with $\nu = 1$.

The third sum is calculated as

$$\frac{\partial}{\partial x}[x F_0(x,\theta)] = \frac{1 - x\cos\theta}{(1+x^2-2x\cos\theta)^{3/2}} = \sum_{l=0}^{\infty} [x D_l'(x) + D_l(x)] P_l(\cos\theta). \tag{A.4}$$

Consider now the sum

$$F(x,\theta) = \sum_{l=1}^{\infty} \frac{l}{l+1} x^{l+1} P_l^1(\cos\theta) = -\frac{\partial}{\partial\theta} \sum_{l=1}^{\infty} \frac{l}{l+1} x^{l+1} P_l(\cos\theta), \tag{A.5}$$

where $x < 1$. It is easy to see that

$$\frac{\partial}{\partial x} F(x,\theta) = -x \frac{\partial^2}{\partial x \partial\theta} F_0(x,\theta) = x\sin\theta \frac{\partial}{\partial x} \frac{x}{(1+x^2-2x\cos\theta)^{3/2}}. \tag{A.6}$$

Using equation (A.6) we finally obtain

$$\begin{aligned} F(x,\theta) &= \sin\theta \left[x^2 F_0^3(x,\theta) - \int_0^x F_0^3(t,\theta) t \, \mathrm{d}t \right] \\ &= \frac{x^2 \sin\theta}{(1+x^2-2x\cos\theta)^{3/2}} - \frac{1}{\sin\theta} \left(1 - \frac{1 - x\cos\theta}{(1+x^2-2x\cos\theta)^{1/2}} \right). \end{aligned} \tag{A.7}$$

In equation (A.7) we have used the initial condition $F(0,\theta) = 0$.

Appendix B. Evaluation of the vector potential

For evaluation of the integral equation (32) we consider the explicit expression for the spherical Hankel functions $h_l^{(1)}(z)$ [15],

$$h_l^{(1)}(z) = (-\mathrm{i})^{l+1}\,\mathrm{e}^{\mathrm{i}z} \sum_{k=0}^{l} \left(\frac{\mathrm{i}}{2}\right)^k \frac{(l+k)!}{k!(l-k)!} \frac{1}{z^{k+1}}, \tag{B.1}$$

and assume that $b_l(\lambda) = B_l/\lambda^{l+1}$, where B_l does not depend on λ. This choice of $b_l(\lambda)$ assures that B_l is constant (see below). Inserting equation (B.1) and $b_l(\lambda)$ into equation (32) we obtain

$$B_l \sum_{k=0}^{l} \left(\frac{\mathrm{i}}{2}\right)^k \frac{(l+k)!}{k!(l-k)!} \left(\frac{1-\beta}{\beta\tau}\right)^{k+1} \Im_{k+l+1}(\tau) = -\mathrm{i}^l \left(\frac{vt}{r_0}\right)^l, \tag{B.2}$$

where $\tau = t(1-\beta) > 0$ and

$$\Im_n(\tau) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} \frac{\mathrm{e}^{-\mathrm{i}\lambda\tau}\,\mathrm{d}\lambda}{\lambda^{n+1}} = \frac{1}{n!}\frac{\partial^n}{\partial q^n}\left[\frac{1}{2\pi\mathrm{i}} \int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} \frac{\mathrm{e}^{-\mathrm{i}\lambda\tau}\,\mathrm{d}\lambda}{\lambda - q}\right]_{q=0}. \tag{B.3}$$

Here $\mathrm{Im}\,q < \sigma$. The integral within the square brackets according to Kochi's theorem and at $\tau > 0$ is equal to $-\mathrm{e}^{-\mathrm{i}q\tau}$. Therefore $\Im_n(\tau) = -(-\mathrm{i}\tau)^n/n!$. Inserting this function into equation (B.2) we arrive at equation (33) (see, e.g., [15]). The complete solution is obtained by inserting equation (33) into equation (31) and evaluating the contour integral as it was done above.

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